Math 206A Lecture 5 Notes

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1 Carathèodory's Theorems and Weak Tverberg's Theorem

1.1 Geometric theorems of Carathèodory

Theorem 1.1 (Bárány). *For every d, there exists a constant $\alpha_d > 0$ such that for every $Z = \{z_1, \dots, z_n\} \subseteq \mathbb{R}^d$, there exists $x \in \mathbb{R}^d$ such that $x \in \operatorname{Conv}(Z_I)$, $|I| = d+1$ for at least $\alpha_d \binom{n}{d+1}$ subsets I.*

To prove this, we'll need some lemmas, all of which are interesting in their own right.

Theorem 1.2 (Carathèodory). *Let $Z = \{z_1, \dots, z_n\} \subseteq \mathbb{R}^d$ with $x \in \operatorname{Conv}(Z)$. Then there exists $I \subseteq [n]$ with $|I| = d+1$ such that $x \in \operatorname{Conv}(Z_I)$.*

Proof. By induction. Fix a vertex v, and use induction to triangulate all facets. Take cones over all simplices in the facets. □

Here is a result which uses an analogue of infinite descent, but in geometry.

Theorem 1.3 (Galloi-Sylvester). *For all $X = \{x_1, \dots, x_n\}$ with the x_i not all on a line, there exist i, j such that the line $(x_i x_j)$ has no other x_r.*

Proof. Let

$$\gamma := \min_{(r,i,j) \text{ distinct}} \operatorname{dist}(x_r, (x_i x_j)).$$

Proceed by contradiction. □

Theorem 1.4 (colorful Carathèodory). *Let $X_1, \dots, X_{d+1} \subseteq \mathbb{R}^d$ be finite sets with $0 \in \operatorname{Conv}(X_i)$ for all i. Then there exist $x_1 \in X_1, x_2 \in X_2, \dots, x_{d+1} \in X_{d+1}$ such that $0 \in \operatorname{Conv}(\{x_1, \dots, x_{d+1}\})$.*

Proof. By contradiction. Let γ be the minimum distance between a colorful simplex and the origin, where the colorful simplexes are the ones formed by x_i. Note that $\gamma > 0$. Let u minimize this distance. The hyperplane H which contains u contains all the x_i except x_1 (wlog). Then there exists $x' \in X_1$ on the other side of H from x_1, otherwise we could not have $0 \in \mathrm{Conv}(X_1)$. Then the distance between 0 and x_1' is smaller than γ, which is a contradiction.

If $u = x_2$ (instead of lying on a facet, it lies on a corner), then there exist $x_i', i \neq 2$ on the other side of the perpendicular hyperplane separating 0 and x_2. Then the distance to the convex hull of $\{x_2\} \cup \{x_i' : i \neq 2\}$ is smaller than γ, which is a contradiction. $\square$

1.2 Weak Tverberg's theorem

Theorem 1.5 (weak Tverberg)**.** *Let $r, d \in \mathbb{N}$. For every $n \geq (r-1)(d+1)^2 + 1$ and $x_1, \ldots, x_n \in \mathbb{R}^d$, there exist $I_1, \ldots, I_r \subseteq [n]$ with $I_i \cap I_j = \varnothing$ such that $\bigcap_{i=1}^r \mathrm{Conv}(X_{I_i}) \neq \varnothing$.*

Proof. Let $k := (r-1)(d+1)$ and $s := n - k$. Observe that every $(d+1)$ subsets of size s have a common point; this is because $k(d+1) < k(d+1) + 1 = n$. By Helly's theorem, there exists $z \in \mathbb{R}^d$ such that $z \in \mathrm{Conv}(X_I)$ for all $|I| \geq s$. So $z \in \mathrm{Conv}(X)$, so by Carathèodory's theorem, there is some $Y_1 \subseteq X$ with $|Y_1| = d+1$ such that $z \in \mathrm{Conv}(Y_1)$. Then $z \in \mathrm{Conv}(X \setminus Y_1)$, so we can get $Y_2 \subseteq X \setminus Y_1$ such that $|Y_2| = d+1$ and $z \in \mathrm{Conv}(Y_2)$. Continue this to get $I_1 = Y_1, \ldots, I_r = Y_r$, which is what we wanted. $\square$

Remark 1.1. The actual Tverberg's theorem is the same but without the power of 2 on the $d+1$ term.